The classifying space of the 1+1 dimensional G-cobordism category

Carlos Segovia

Abstract. The 1+1 G-cobordism category, with G a finite group, is important in the construction of G-topological field theories which are completely determined by a G-Frobenius algebra, see [**MS06, Tur10, Kau03**]. We give a description of the classifying space of this category generalizing the work presented by Ulrike Tillmann in [**Til96**]. Moreover, we compute the connected components and the fundamental group of this classifying space and we give a complete description of the classifying spaces of some important subcategories. Finally, we present some relations between the rank of the fundamental group of the G-cobordism category and the number of subgroups of the group G.

Introduction

Nowadays the study of the classifying space of cobordism categories is rapidly gaining importance due to the proof by Madsen and Weiss in [**MW07**], of Mumford's conjeture about the stable cohomology of the moduli space of Riemann surfaces. Initially, the cobordism category was introduced by Segal in [**Seg74**], for the study of conformal field theories in dimension 1+1 (where 1 represents the dimension of the objects and 1+1=2 the dimension of the morphisms of the cobordism category). Tillmann in [**Til96**], was the first who provided a study of its classifying space and in collaboration with Galatius, Madsen and Weiss culminated in the calculation of the homotopy type of the cobordism category in any dimension, see [**GMTW**]. They showed that in every dimension the classifying space of the cobordism category has the homotopy type of the infinite loop space of a certain Thom-spectra. For G a finite group, the G-cobordism category was introduced by Turaev in [**Tur10**], with a homotopical version given by a background space with base point, that in our case is the classifying space BG. The definition we use in this article for the G-cobordism category was given by Segal and Moore in [**MS06**] and by Kaufmann in [**Kau03**]. Let $\mathscr{S}^G$ be the G-cobordism category in dimension $1+1$, the principal result of this article is the calculation of the homotopy type of the classifying space

$$B\mathscr{S}^G \simeq \frac{G}{[G,G]} \times X_G \times T^{r(G)}\,, \tag{0.1}$$

where X_G is the homotopy fiber of the classifying map of a certain functor and $T^{r(G)}$ is the direct product of $r(G)$-circles, with $r(G)$ a positive integer which depends on G. In addition, we prove that the connected components are parameterized by the elements of the abelianization of G, thus $\pi_0(\mathscr{S}^G) \cong G/[G,G]$, and the fundamental group satisfies the isomorphism $\pi_1(\mathscr{S}^G) \cong \mathbb{Z}^{r(G)}$. Calculations of the number $r(G)$ for some groups G are presented in table (5.5) which provide unknown relations with geometric group theory through an approximation of the number of subgroups of the group G. Some future extensions of our work consist to substitute the group G by a groupoid $\mathcal{G}$, see [**Pha10**], and to consider an arbitrary space as the background space in the homotopical version, see [**Tur10, BT99**].

This article is organized as follows. In **section 1** we define the G-cobordism category in terms of the elementary components given by the G-principal bundles over the circle, the pair of pants, the cylinder and the disc. In **section 2** we prove for the G-cobordism category, that the connected components of the classifying space are parameterized by the elements of the abelianization of the group and, in addition,

we study the classifying space of some important subcategories. In **section 3** we prove that the category of fractions of the G-cobordism category, can be obtained by just inverting the sphere; this fact permit us to reinterpret its fundamental group and allow us to prove that $\pi_1(\mathscr{S}^G) \cong \mathbb{Z}^{r(G)}$. In **section 4** we focus on the proof that the classifying space of a symmetric monoidal category is an infinite loop space and we give the proof of the homotopy defined in equation (0.1). Finally, in **section 5** we report some results which include a set of equations which determine completely a morphism in the G-cobordism category (this equations were proved in [**Seg11**]). This equations depend only on the axioms of a G-Frobenius algebra [**MS06**]. We implement these equations in MATLAB [**MAT**] to obtain the data of table (5.5); this table allows us to obtain some relations with other fields in mathematics, that were discovered with the on-line encyclopedia of integer sequences [**oei**].

I would like to thank to the three persons which helped me throughout my Ph.D. studies, I hold their teachings, in math and in the life, in high regard. They are Ana González and my two teachers Ernesto Lupercio and Bernardo Uribe.

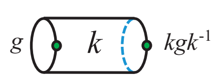
The results of this article are part of the author's Ph.D. thesis under the direction of Dr. Ernesto Lupercio at the Math Department of the CINVESTAV.

1. Definition of the G-cobordism category

For G a finite group, the G-cobordism category in dimension $1+1$, denoted by $\mathscr{S}^G$, has principal G-bundles over circles (up to homeomorphisms) as objects and principal G-bundles over surfaces (up to homeomorphisms) as morphisms. The principal G-bundles over the circle are based maps $P \xrightarrow{\pi} S^1$, with P and S^1 based spaces and π a based map. P_1 and P_2 are homeomorphic if there exists a homeomorphism $P^1 \longrightarrow P^2$ which is a map of principal G-bundles. Thus they are in correspondence with the elements of the group G (this elements are called G-circles). Therefore the objects of the category $\mathscr{S}^G$ are described by sequences $(g_1, g_2, ..., g_n)$ of elements in G except for some of them that are the empty set $\emptyset$; the order of the sequences goes downwards in our pictures. For the morphisms of $\mathscr{S}^G$ consider $\overline{g} := (g_1, g_2, ..., g_n)$ and $\overline{h} := (h_1, h_2, ..., h_m)$ sequences as before, denote $\bigsqcup_i P_{g_i}$ and $\bigsqcup_j P_{h_j}$ their corresponding total spaces; a morphism from $\overline{g}$ to $\overline{h}$ is composed by a cobordism M of the based spaces and a principal G-bundle $P \xrightarrow{\pi} M$ with $P|_{\partial_{in}} = \bigsqcup_i P_{g_i}$ and $P|_{\partial_{out}} = \bigsqcup_j P_{h_j}$. Two cobordisms are identified if there is a homeomorphism $M \longrightarrow M'$ of principal G-bundles fixing the boundary.

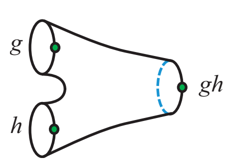
For the morphisms of $\mathscr{S}^G$ we do not have explicit expression, but we can study them through their decomposition in elementary parts. The elementary parts are the principal G-bundles over the cylinder, the pair of pants and the disk; the description is as follows:

- for $g, h \in G$, the morphisms from g to h (with base space the cylinder) are in one-to-one correspondence with the elements of the set $\{k : h = kgk^{-1}\}$ up to the identification[1] $k \sim h^n k g^m$, where $n, m \in \mathbb{Z}$, and a typical element is

  .

  This correspondence is given by means of the homotopy lifting property applied to the base space, a cylinder, with starting point P_g;
- the pair of pants has the homotopy type of $S^1 \vee S^1$ and since G is finite, the set of principal G-bundles over the pair of pants (with base point the critical point of index one) is in bijection with the group homomorphisms from the fundamental group of $S^1 \vee S^1$ to G, i.e. with $G \times G$. A basic element is

[1]This identification is given by the action of the Dehn twists.

and any other principal G-bundle over the pair of pants is obtained by composition with principal G-bundles of the cylinder; and
- the disk which is contractile, therefore it has only one principal G-bundle over it which is trivial.

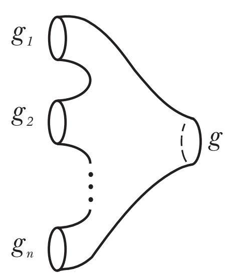
This elements are called G-cylinders, G-pair of pants and the disc respectively. Every G-cobordism is constructed by composition of these elements, together with the corresponding elements with the reverse orientation, restricted up to some constrains which are given in section 5. Finally, the composition of two morphisms is done in such a way that the base points match.

2. Analysis of subcategories

We start this section with the proof that the connected components of the classifying space of $\mathscr{S}^G$ is the abelianization $G/[G,G]$. This reduces the calculation of the classifying space of $\mathscr{S}^G$ to the connected component of the empty 1-manifold $\emptyset$ or to the connected component of the trivial G-circle (the one associated to the identity $e \in G$). The rest of this section is devoted to the study of the classifying space of some smaller subcategories of $\mathscr{S}^G$.

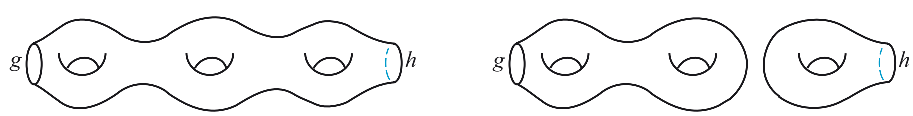
PROPOSITION 2.1. *The connected components of the classifying space $B\mathscr{S}^G$ are in bijection with the elements of the abelianization of the group, i.e. with the quotient $G/[G,G]$.*

PROOF. First, we note that the empty set and the trivial G-circle are connected through the disc. Therefore, we can restrict the category $\mathscr{S}^G$ to the full subcategory with the same objects of $\mathscr{S}^G$ except for the empty set $\emptyset$; hence every object of this subcategory is a sequence $(g_1, g_2, ..., g_n)$ of elements in G which is connected to the product $g = \prod_i^n g_i$ by the G-pair of pants with multiple legs



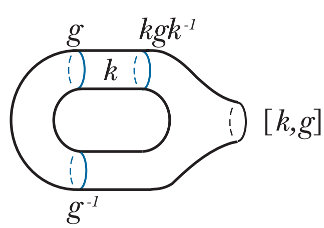
(2.1)

Furthermore, the G-cobordisms with empty boundary are superfluous to determine the connected components, and consequently it is enough to consider the morphisms of the forms



(2.2)

with $g, h \in G$. Finally, since every handle has as boundary a commutator,



and the figures in (2.2) are compositions of handles with G-pair of pants of the from (2.1), consequently, the G-circles associated to $g, h \in G$ are connected in $\mathscr{S}^G$ if and only if they differ by an element of the commutator group. $\square$

Corollary 2.2. *The classifying space $B\mathscr{S}^G$ is of the homotopy type of the product of $G/[G,G]$ with the connected component of the empty 1-manifold $\emptyset$*[2].

Proof. The category $\mathscr{S}^G$ is a symmetric monoidal category with braiding the twist of cylinders and monoidal structure given by the disjoint union and the empty set. We prove in Theorem 4.3 that the classifying space of a symmetric monoidal category is an H-space. Moreover, it is well known that if an H-space has a group structure in the connected components, compatible with the product of the H-space, then every pair of connected components are homotopy equivalent by multiplication by an element (an H-space with this property is called a *grouplike*, see [**May74**]). Since the connected components $\pi_0(\mathscr{S}^G)$ have the group structure $G/[G,G]$, which is compatible with the monoidal structure, i.e. we have the identity $[g_1 \sqcup g_2] = [g_1 g_2]$ with $[g]$ the connected component of the G-circle associated to $g \in G$. Therefore we can decompose the classifying space $B\mathscr{S}^G$ as a product of $G/[G,G]$ with the connected component of the unit, which in this case is the point given by the empty set $\emptyset$. $\square$

Denote $\mathscr{S}_0^G$ the full subcategory of $\mathscr{S}^G$ with only one object given by the empty set.

Proposition 2.3. *The classifying space $B\mathscr{S}_0^G$ is homotopic to the infinite dimensional torus T^∞.*

Proof. This category is endowed with the structure of an abelian monoid, infinitely generated, without torsion and hence of the form $\mathbb{N}^\infty$. Since the classifying map of the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ is a homotopy equivalence and since the classifying space of $\mathbb{Z}$ is the circle, then $B\mathscr{S}_0^G \simeq T^\infty$ where the infinite torus has the direct limit topology induced from the finite dimensional subtori. $\square$

Remark 2.4. From here, we restrict every proper subcategory to the condition that every object is connected to the empty set . To recover the classifying space of the complete subcategories we just make the product with the abelianization of G.

Now, we denote by $\mathscr{S}_{>0}^G$ the subcategory of $\mathscr{S}^G$ with the same objects of $\mathscr{S}^G$ except for the empty set and where each connected component of every morphism has non empty initial boundary and non empty final boundary.

Proposition 2.5. *The classifying space $B\mathscr{S}_{>0}^G$ is homotopy equivalent to the Borel construction $T^{r(G)} \times_G EG$, with $T^{r(G)} := \prod_{i=1}^{r(G)} S^1$ and $r(G)$ a positive integer.*

In order to prove this result we need to consider some smaller subcategories. Let $\mathscr{S}_1^G$ be the subcategory of $\mathscr{S}^G$ where each object is a G-circle and the morphisms are connected G-cobordisms. Let $\mathscr{S}_e^G$ be the full subcategory of $\mathscr{S}_1^G$ with only one object given by the G-circle with trivial principal G-bundle. We recall from [**Seg68**] that a natural transformation between two functors translate into a homotopy equivalence between the classifying maps of the functors. Moreover, the classifying space of a semi-direct product is the Borel construction, see [**Seg11**]. Therefore the Proposition 2.5 is implied by the following results.

Proposition 2.6. *The inclusion functor $\mathscr{S}_e^G \hookrightarrow \mathscr{S}_1^G$ is a homotopy equivalence in classifying spaces.*

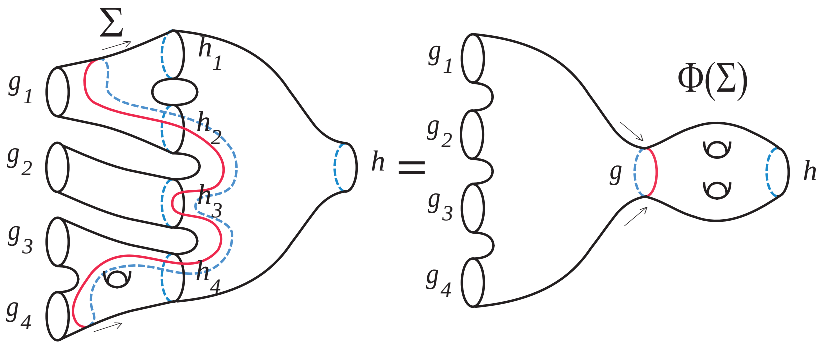
Proposition 2.7. *There is an isomorphism of $\mathscr{S}_e^G$ with a semi-direct product $\mathbb{N}^{r(G)} \rtimes G$ for $r(G)$ a positive integer.*

Theorem 2.8. *The inclusion functor $\mathscr{S}_1^G \longrightarrow \mathscr{S}^G$ has a left adjoint Φ.*

Proof. We proceed to define the functor $\Phi : \mathscr{S}_{>0}^G \longrightarrow \mathscr{S}_1^G$. On objects this functor is settled by multiplication $(g_1, g_2, ..., g_n) \longmapsto g = \prod_i g_i$ and for morphisms we take the following construction: take Σ a G-cobordism in $\mathscr{S}_{>0}^G$ from (g_i) to (h_j), $g = \prod_i g_i$ and $h = \prod_j h_j$; then we compose Σ with a G-pair of pants with multiple legs with one exit and the same entries that the exits of Σ (of the form (2.1)); subsequently, for the resulting connected cobordism by Cerf theory [**Cer70**], we can find a Morse function for a representant of Σ, such that there exists $t \in [0,1]$ with the property that the inverse image of $[0,t]$

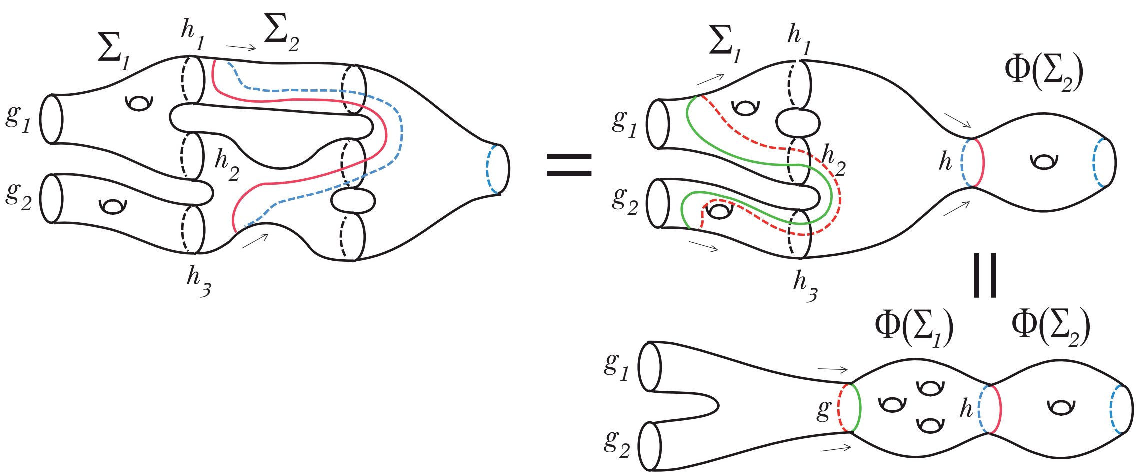
[2] We recall that the empty 1-manifold and the trivial G-cylinder are connected through the disc.

is a pair of pants with multiple legs, of the form (2.1), and the inverse image of t is a circle; finally, we take $\Phi(\Sigma)$ as the class in $\mathscr{S}_1^G$ of the pre-image of $[t, 1]$ by the last Morse function, see the following figure



(2.3)

The functorial property of Φ is illustrated in the following figure for a particular case which admits extension in a natural way:



Eventually, the figure (2.3) expresses the adjointness of Φ by the commutativity of the diagram

$$\begin{tikzcd} (g_i) \arrow[r, "p_{(g_i)}"] \arrow[d, "\Sigma"'] & g \arrow[d, "\Phi(\Sigma)"] \\ (h_j) \arrow[r, "p_{(h_j)}"'] & h, \end{tikzcd}$$

where $p_{(g_i)}$ and $p_{(h_j)}$ are G-pair of pants of the form (2.1). □

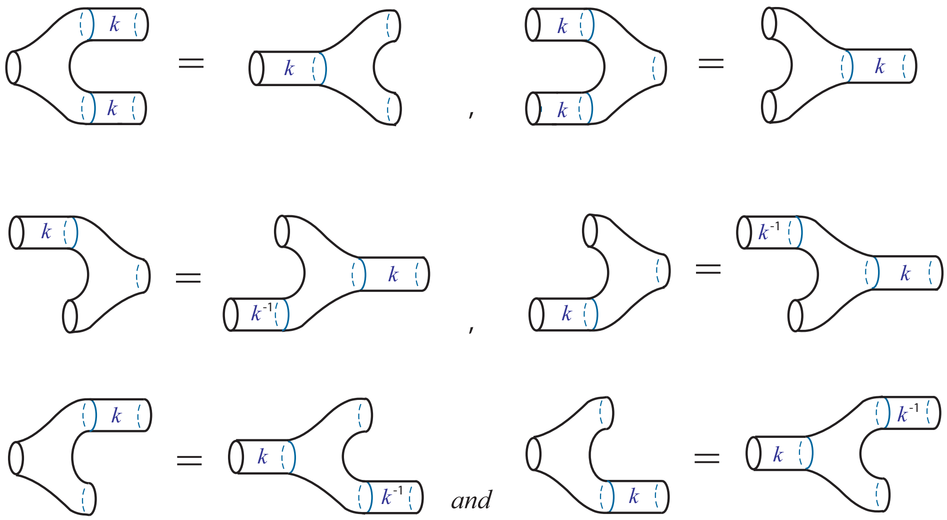
There is a subcategory $\mathscr{S}_b^G$ of $\mathscr{S}^G$ similar to $\mathscr{S}_{>0}^G$. This category has the same objects as $\mathscr{S}^G$ and each connected component of every morphism has non empty final boundary[3]. Similarly, we can define a functor $\Phi : \mathscr{S}_b^G \longrightarrow \mathscr{S}_1^G$ and we can state the following result.

Proposition 2.9. *The classifying space of $\mathscr{S}_b^G$ is homotopy equivalent to the Borel construction $T^{r(G)} \times_G EG$.*

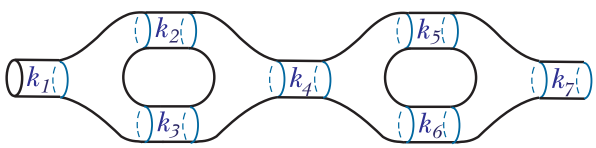
Now, we end this section with the proof of the Propositions 2.6 and 2.7.

[3]Note that by vacuity the empty manifold satisfies to be the identity of the empty set.

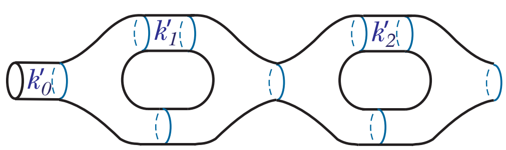
Proof of Proposition 2.6. We have the following relations in the G-cobordism category $\mathscr{S}^G$

(2.4)


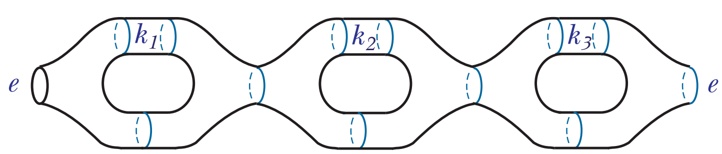
These properties imply that any morphism in $\mathscr{S}_e^G$ of the form



can be reduced by a successive application of these relations to the following



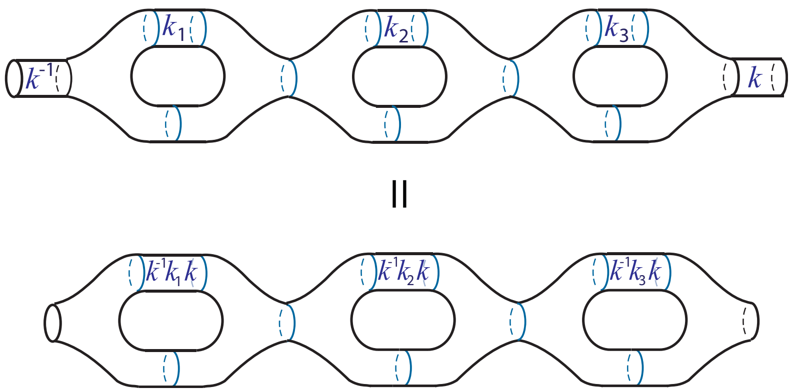
where $k'_0 = k_7k_6k_4k_3k_1$, $k'_1 = k_7k_6k_4k_2{k_3}^{-1}{k_4}^{-1}{k_6}^{-1}{k_7}^{-1}$ and $k'_2 = k_7k_5{k_6}^{-1}{k_7}^{-1}$. And it is straightforward to consider the monoid composed by elements of the form

(2.5)


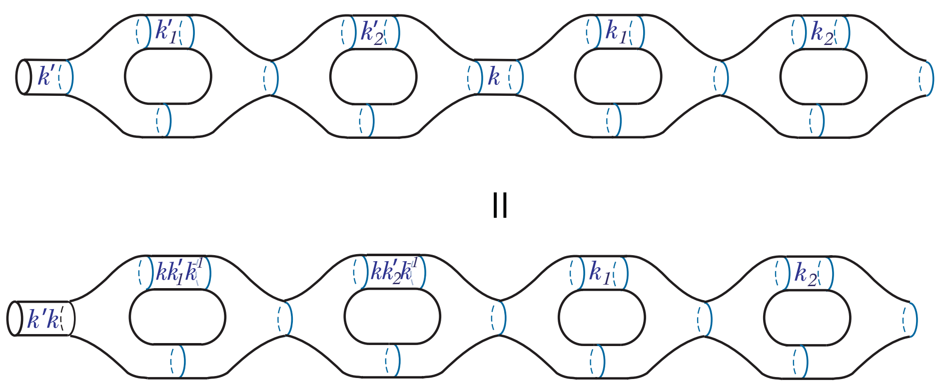
where we depict in (2.5) a morphism of genus three, but with this we exemplified every morphism of any genus. We also note that in picture (2.5) the only nontrivial G-cylinders[4] are the ones corresponding to the elements $k_1, k_2, k_3 \in G$. To construct the semi-direct product we need to define the action, which for

[4]Recall that the trivial G-cylinder is the associated to the identity $e \in G$.

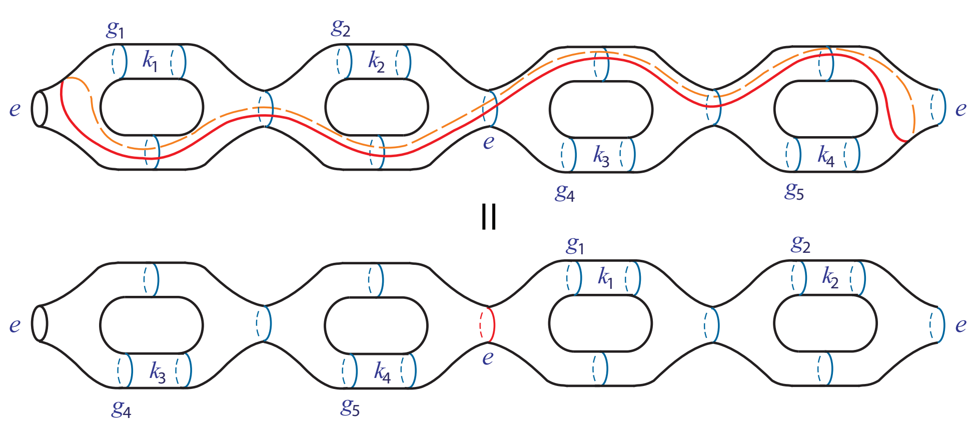
$k \in G$ is defined by conjugation as follows



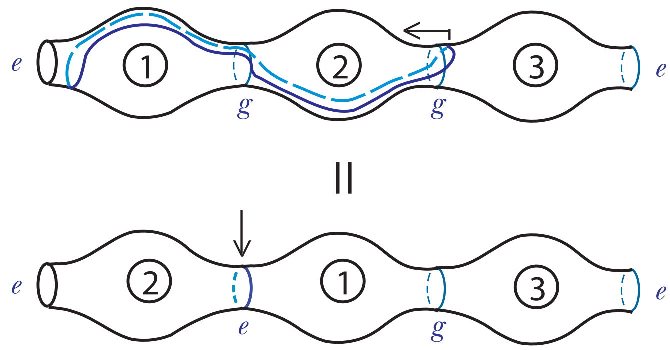
Thus the composition in $\mathscr{S}_e^G$ can be described by



This monoid is abelian and finitely generated, which are exemplified in the following figures respectively,



and

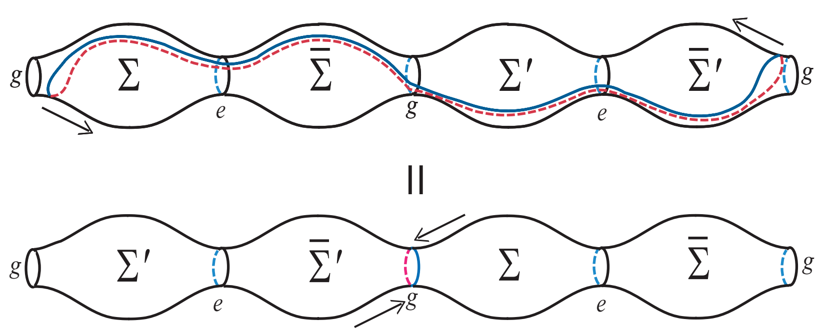


Moreover, this monoid does not have torsion because the genus map, which associates to any G-cobordism the genus of the base space, has trivial kernel. Consequently, this monoid is the direct sum $\mathbb{N}^{r(G)}$ and the category $\mathscr{S}_e^G$ is isomorphic to the semi-direct product $\mathbb{N}^{r(G)} \rtimes G$.

□

Proof of Proposition 2.7. For a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ and y an object of $\mathcal{D}$, we define the category $y \diagdown F$ with objects pairs (v, x) with x an object in $\mathcal{C}$ and $v : y \longrightarrow F(x)$ a morphism in $\mathcal{D}$; a morphism $f : (v, x) \longrightarrow (v', x')$ is given by a morphism $f : x \longrightarrow x'$ in $\mathcal{C}$ with $F(f) \circ v = v'$. A celebrated theorem of Quillen, see [**Qui73**], states that for small categories the classifying map of F is a homotopy equivalence if every category $y \diagdown F$ has contractible classifying space, this is the case when the category $y \diagdown F$ is filtrated. The category $y \diagdown F$ is filtrated if we have the following two conditions: first for any pair of objects $(v, x), (v', x')$ in $y \diagdown F$, we can make the maps v and v' equal in $\mathcal{D}$ through two morphism $u : x \longrightarrow x'', u' : x' \longrightarrow x''$ in C; and second for any object (v, x) and two parallel morphisms $u, u' : x \longrightarrow x'$ in C, there is a third morphism w in $\mathcal{C}$ such that $w \circ u = w \circ u'$. For the first condition, we take two objects in $g \diagdown i$ as follows

and we get the next equation



where $\overline{\Sigma}$, $\overline{\Sigma'}$ are Σ, Σ' but with the opposite direction. The second condition of a filtrated category is satisfied since the category $\mathscr{S}_1^G$ satisfies cancelation law. □

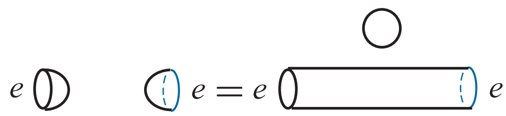
3. The fundamental group

Now we make a further analysis of the fundamental group of the whole category $\mathscr{S}^G$. The category $\mathscr{S}^G$ is essentially the union of the categories $\mathscr{S}_0^G$ and $\mathscr{S}_{>0}^G$ with the disc from the empty 1-manifold to the trivial G-circle and vice versa as additional generators. We prove in the last section that the connected components satisfy the identity $\pi_0(\mathscr{S}^G) = G/[G, G]$. By Proposition 2.3 and Proposition 2.5 we expect

a big fundamental group or a semi-direct product, but we have $\pi_1(B\mathscr{S}^G) \cong \mathbb{Z}^{r(G)}$. We prove this result in this section.

Let $\widehat{\mathscr{S}^G}$ be the quotient category $\mathscr{S}^G$ with the equivalent relation generated by the following identification,

(3.1)



with this equivalence relation we refer that two G-cobordisms are identified if we can get one from the other through a finite number of steps, given by elimination of a sphere with the trivial G-cylinder. Denote by $\mathscr{G}^G = \mathscr{S}^G[\mathscr{S}^{G^{-1}}]$ the category of fractions obtained by inverting all the morphisms of $\mathscr{S}^G$, see [**GZ67, GM03**]. We denote by $\mathcal{I}$ the subset of morphisms of $\mathscr{S}^G$ given by disjoint union of G-surfaces (G-cobordisms with empty boundary) with trivial G-cylinders.

Proposition 3.1. *There is an isomorphism between $\mathscr{G}^G$ and $\widehat{\mathscr{S}^G}[\mathcal{I}^{-1}]$.*

Proof. The category of fractions has an easy description when we have a localizing set, see [**GM03, Qui73**]. For $\mathcal{C}$ a category and J a subset of morphisms of $\mathcal{C}$ we say that J is a localizing set if we have the following properties:

1) the set J contains the identities, i.e. $1_x \in J$ for any object of $\mathcal{C}$, the set J is closed under composition, i.e. $s \circ t \in J$ for any $s, t \in J$ whenever the composition is defined;
2) for any morphism f of $\mathcal{C}$, $s \in J$ with common end, there exist morphisms g in $\mathcal{C}$ and $t \in J$ such that the following square

(3.2)
$$\begin{tikzcd} w \arrow[r, "g"] \arrow[d, "t"'] & z \arrow[d, "s"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

is commutative;
3) let f, g be two morphisms from x to y; the existence of $s \in J$ with $s \circ f = s \circ g$ is equivalent to the existence of $t \in J$ with $f \circ t = g \circ t$.

An important result of having a localizing set is the simplicity under which one can write the category of fractions, see [**GM03**]. Let J be a localizing subset of the morphisms in a category $\mathcal{C}$. The category of fractions $\mathcal{C}[J^{-1}]$ can be described as follows: the objects of $\mathcal{C}[J^{-1}]$ are the same as in $\mathcal{C}$ and one morphism $x \longrightarrow y$ in $\mathcal{C}[J^{-1}]$ is a class of "roofs", i.e. of diagrams (s, f), in $\mathcal{C}$ of the form

$$\begin{tikzcd} & x' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ x & & y\,, \end{tikzcd}$$

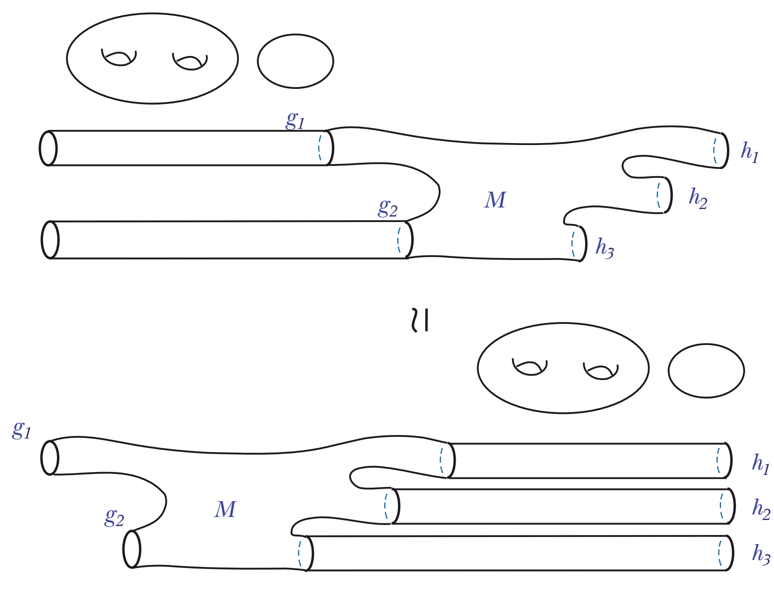
where $s \in S$ and f is a morphism in $\mathcal{C}$, and two roofs are equivalent $(s, f) \sim (t, g)$ if and only if there exists a third roof (r, h) forming a commutative diagram of the form

(3.3)
$$\begin{tikzcd} & & x''' \arrow[dl, "r"'] \arrow[dr, "h"] & & \\ & x' \arrow[dl, "s"'] \arrow[drrr, "f"', near end] & & x'' \arrow[dlll, "t", near end] \arrow[dr, "g"] & \\ x & & & & y\,. \end{tikzcd}$$

Moreover, the identity morphism $1 : x \longrightarrow x$ is the class of the roof $(1_x, 1_x)$ and the composition two roofs (s, f) and (t, g), is the class of the roof $(s \circ t', g \circ f')$ obtained by using the first square in (3.2) as follows,

$$
\begin{tikzcd}
 & & x'' \arrow[dl, "t'"'] \arrow[dr, "f'"] & & & & & x'' \arrow[dl, "st'"'] \arrow[dr, "gf'"] & \\
 & x' \arrow[dl, "s"'] \arrow[dr, "f"] & & y' \arrow[dl, "t"'] \arrow[dr, "g"] & & & & & \\
x & & y & & z & \Longrightarrow & x & & y\,.
\end{tikzcd}
$$

The following picture shows that the set $\mathcal{I}$ is a localizing set inside $\widehat{\mathscr{S}^G}$:

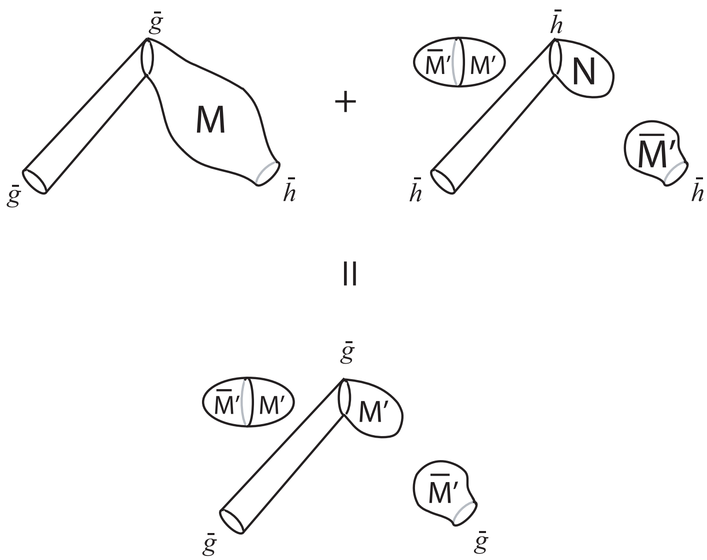


The universal properties of a category of fractions and of a quotient category, assure the existence of unique functors $E : \widehat{\mathscr{S}^G}[\mathcal{I}^{-1}] \longrightarrow \mathscr{G}^G$ and $F : \widehat{\mathscr{S}^G} \longrightarrow \mathscr{G}^G$ such that the following diagrams commute

$$
\begin{tikzcd}
\widehat{\mathscr{S}^G} \arrow[r, "J'"] \arrow[dr, "F"'] & \widehat{\mathscr{S}^G}[\mathcal{I}^{-1}] \arrow[d, "E"] \\
 & \mathscr{G}^G
\end{tikzcd}
,\qquad
\begin{tikzcd}
\mathscr{S}^G \arrow[r, "P"] \arrow[dr, "J"'] & \widehat{\mathscr{S}^G} \arrow[d, "F"] \\
 & \mathscr{G}^G
\end{tikzcd}
.
$$

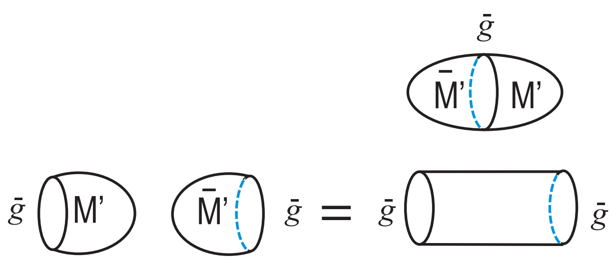
By the representation of "roofs" of the category of fractions, we can see that the composition $L := J' \circ P$ inverts any morphism of $\mathscr{S}^G$. For this, take M a G-cobordism from $\overline{g} := (g_1, ..., g_n)$ to $\overline{h} := (h_1, ..., h_m)$ and let N be a (connected) G-cobordism from $\overline{h}$ to the empty set, denote by M' the composition of M

with N and $\overline{M}'$ denotes M' with the reverse orientation. Thus the following composition

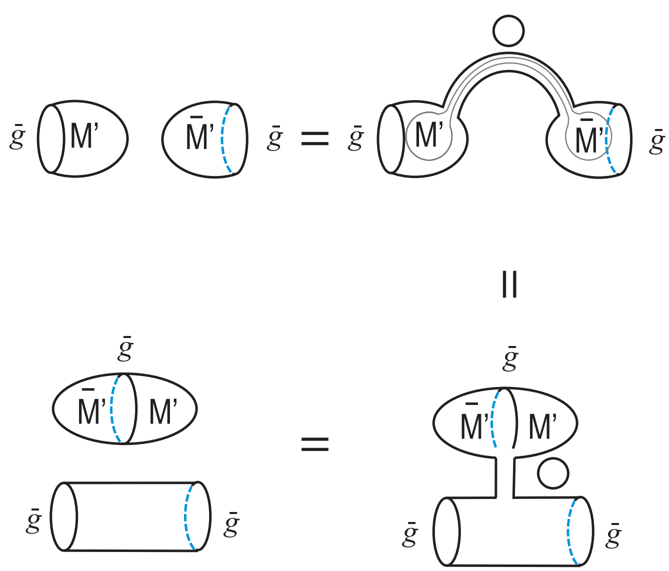


.

is equivalent to the identity morphism, see (3.3), if we have the following identification



For this we use the identification (3.1),



As a consequence, there is a unique functor $D : \mathscr{G}^G \longrightarrow \widehat{\mathscr{S}^G}[\mathcal{I}^{-1}]$ such that the diagram

$$\begin{tikzcd} \mathscr{S}^G \arrow[r, "J"] \arrow[dr, "L"'] & \mathscr{G}^G \arrow[d, "D"] \\ & \widehat{\mathscr{S}^G}[\mathcal{I}^{-1}] \end{tikzcd}$$

is commutative. Therefore we have the sequence of identities

$$E \circ D \circ J = E \circ L = E \circ J' \circ P = F \circ P = J\,,$$

and by the universal property of the category of fractions we conclude that the composition $E \circ D$ is the identity functor; reciprocally, since the functor P is an epimorphism in the category of small categories[5], then the identity $J' \circ P = L = D \circ J = D \circ F \circ P$ implies that $J' = D \circ F$, and consequently $D \circ E \circ J' = D \circ F = J'$ and by the universal property of the category of fractions we have $D \circ E = 1$. □

As a consequence of the last proposition the fundamental group of the classifying space of $\mathscr{S}^G$ is abelian. This because we can take as base point the empty 1-manifold. Moreover, the fundamental group is independent of the base point, since two different connected components of the classifying space of $\mathscr{S}^G$ have the same homotopy type.

THEOREM 3.2. *The fundamental group $\pi_1(B\mathscr{S}^G)$ is isomorphic to the direct product $\mathbb{Z}^{r(G)}$.*

PROOF. We know by [**Qui73**] that the fundamental group of the classifying space of a small category, based on an object of the category, is isomorphic with the restriction of the category of fractions to this object, see [**GZ67**]. Let $\mathscr{G}^G_e$ be the group defined as the full subcategory of $\mathscr{G}^G = \mathscr{S}^G[\mathscr{S}^{G^{-1}}]$ with only one object given by the trivial G-bundle. By Proposition 3.1 this group is composed by roofs of the form

$$\begin{tikzcd} & e \arrow[dl, "z_-"'] \arrow[dr, "z_+"] & \\ e & & e\,, \end{tikzcd}$$

where z_+ and z_- are the positive and negative part, given by G-surfaces (with empty boundary), inside $\widehat{\mathscr{S}^G}$. We recall the monoid defined in Proposition 2.6, which is composed by objects of the form (2.5). We proved that this monoid is isomorphic to the direct product $\mathbb{N}^{r(G)}$. There exists a morphism of monoids $j : \mathbb{N}^{r(G)} \longrightarrow \mathscr{G}^G_e$, defined for a connected G-cobordism Σ, which starts and ends in the trivial G-bundle by

$$\begin{tikzcd} & e \arrow[dl, "1"'] \arrow[dr, "\Sigma"] & \\ e & & e\,. \end{tikzcd}$$

The unique category $e\diagdown j$, defined in the proof of Proposition 2.7, is filtrated. The first axiom of a filtrated category is satisfied by the commutativity of the diagram

$$\begin{tikzcd} e \arrow[r, "z_+ + z_-"] \arrow[d, "z'_+ + z'_-"'] & e \arrow[d, "-z_- + z'_+"] \\ e \arrow[r, "-z'_- + z_+"'] & e\,, \end{tikzcd}$$

where $-z_- + z'_+$ and $-z'_- + z_+$ are inside the initial monoid of j, by adding and subtracting spheres. The second axiom of a filtrated category, is a consequence of the fact that for any morphism in $\mathscr{G}^G_e$ we can cancel its negative part, then we can make it a connected morphism (adding and subtracting spheres) and for the resulting morphism, we can take it with the opposite direction. □

[5]An epimorphism in the category of small categories is a full and an essentially surjective functor.

4. Multiplicative structure

Firstly, we start with an analysis of the infinite loop structure of the classifying space of a symmetric monoidal category and lastly, we conclude with the homotopy type of the classifying space of the whole category $\mathscr{S}^G$.

A space Y is an *infinite loop space* if there is a sequence of spaces $Y_0, Y_1, Y_2 ...$, denoted by (Y_i), with $Y_0 = Y$ and with weak equivalences[6]

$$Y_n \xrightarrow{\simeq} \Omega Y_{n+1} \,.$$

For infinite loop spaces (Y_i) and (Z_i), a map of infinite loop spaces is given by a sequence of maps $f_i : Y_i \longrightarrow Z_i$ with the following commutative diagrams

$$\begin{tikzcd} Y_i \arrow[r] \arrow[d, "f_i"'] & \Omega Y_{i+1} \arrow[d, "\Omega f_{i+1}"] \\ Z_i \arrow[r] & \Omega Z_{i+1}\,. \end{tikzcd}$$

PROPOSITION 4.1. *For a map of infinite loop spaces $(Y_i) \xrightarrow{f_i} (Z_i)$, the homotopy fibers (F_{f_i}) form an infinite loop space.*

PROOF. Given a map $f : Y \longrightarrow Z$ we can extend f to a fibration by enlarging its domain to a homotopy equivalent space. This space, denoted by E_f, consists of pairs (y, γ) where $y \in Y$ and $\gamma : I \longrightarrow Z$ is a path in Y (non-based) with $\gamma(0) = f(y)$. We topologize E_f as a subspace of $Y \times Z^I$, where Z^I is the space of maps $I \longrightarrow Z$ (non-based) with the compact-open topology. We can regard Y as the subspace of E_f consisting of pairs (y, γ) with γ the constant map at $f(y)$, and E_f deformation retracts onto this subspace by restricting all the path γ to shorter and shorter initial segments. The map $p : E_f \longrightarrow Z$, which is given by $p(y, \gamma) = \gamma(1)$, is a fibration. The fiber F_f of p is called the homotopy fiber of f. Let PZ be the subspace of Z^I of based maps, where I is given the base point 0. Hence we have the pullback square

$$\begin{tikzcd} F_f \arrow[r] \arrow[d] & PZ \arrow[d, "e_1"] \\ Y \arrow[r, "f"'] & Z\,, \end{tikzcd}$$

where e_1 is the evaluation at 1. Since $\Omega PZ = P\Omega Z$, then the adjointness between the suspension Σ and the loop space Ω functors implies that $F_{\Omega f} = \Omega F_f$. Finally, for $\varphi_i : Y_i \longrightarrow \Omega Y_{i+1}$ and $\psi_i : Z_i \longrightarrow \Omega Z_{i+1}$ the homotopy equivalences $\phi_i : F_{f_i} \longrightarrow \Omega F_{f_{i+1}}$ are given by $(y, \gamma) \longmapsto (\varphi_i(y), \psi_i \gamma)$, and each one satisfies the commutative diagram

$$\begin{tikzcd} F_{f_i} \arrow[r, "\phi_i"] \arrow[d, "v_i"'] & \Omega F_{f_{i+1}} \arrow[d, "\Omega v_{i+1}"] \\ Y_i \arrow[r, "\varphi_i"] \arrow[d, "f_i"'] & \Omega Y_{i+1} \arrow[d, "\Omega f_{i+1}"] \\ Z_i \arrow[r, "\psi_i"'] & \Omega Z_{i+1}\,. \end{tikzcd}$$

By the five lemma we conclude the proposition. $\square$

[6]A weak homotopy equivalence between topological spaces is a continuous map which induces isomorphisms on all homotopy groups.

Proposition 4.2. *For (X_i), (Y_i) infinite loop spaces and $(X_i) \xrightarrow{f_i} (Y_i)$ a map of infinite loop spaces, if there is some $i \geq 1$ and a map $q : \Omega Y_i \longrightarrow \Omega X_i$ with $\Omega f_i \circ q \simeq 1$, then for $X := X_0$, $Y := Y_0$ and $f := f_0$ the next identity follows $X \simeq F \times Y$, with F the homotopy fiber of f.*

Proof. An important result proved by Eckmann and Hilton in [**EH60**], states that for a fibration $F \xrightarrow{j} X \xrightarrow{f} Y$, with $j \simeq 0$, there is a map $d : F \longrightarrow \Omega Y$ such that $m \circ (\Omega f \times d) : \Omega X \times F \longrightarrow \Omega Y$ is a homotopy equivalence (for m the product of loops in ΩY). For continuous maps $f : X \longrightarrow Y$ and $q : \Omega Y \longrightarrow \Omega X$, with $\Omega f \circ q \simeq 1$, we consider the fiber sequence generated by the map f

$$\begin{tikzcd} \cdots \arrow[r] & \Omega X \arrow[r, "\Omega f"] & \Omega Y \arrow[l, bend right, "q"'] \arrow[r, "s"] & F_f \arrow[r] & X \arrow[r, "f"] & Y , \end{tikzcd}$$

since $s \circ \Omega f \simeq 0$ and by assumption we have $\Omega f \circ q \simeq 1$, then $s \simeq s \circ \Omega f \circ q \simeq 0$ for the fibration $\Omega Y \xrightarrow{s} F_f \xrightarrow{j} X$. Consequently, there is a homotopy equivalence $\Omega X \simeq \Omega F_f \times \Omega Y$, where F_f is the homotopy fiber of f. By the definition of an infinite loop space the proposition follows. □

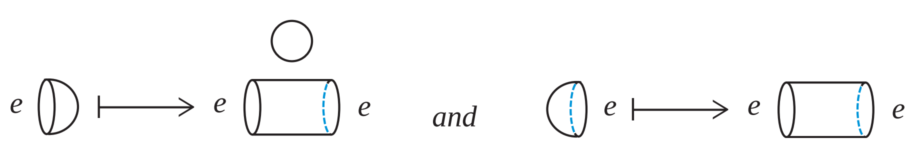
Theorem 4.3. *The classifying space of a symmetric monoidal category, with group structure in the connected components compatible with the monoidal product, has the homotopy type of an infinite loop space.*

Proof. There are machineries which serve as recognizable principles for infinite loop spaces. One of them is the theory of permutative categories, see [**May72, May74**]. We choose another way through the theory of Γ-spaces, see [**Seg74**]. Indeed, there is a generalization of what is called a Γ-category. A Γ-category has associated a classifying space which is a Γ-space $A : \Gamma \longrightarrow \mathcal{T}op$[7], where the space $A(1)$, associated to the set $1 := \{1\}$, is an infinite loop space if and only if the space $A(1)$ has homotopy inverse. For a symmetric monoidal category $\mathcal{M}$, we can construct an associated Γ-category where the classifying space (which is a Γ-space) has $A(1) = B\mathcal{M}$; this space is an infinite loop space when there is a homotopy inverse. The theorem follows since there is a group structure in the connected components compatible with the monoidal product. □

As a consequence of Proposition 4.2, for a composition $\mathcal{N} \overset{j}{\hookrightarrow} \mathcal{M} \xrightarrow{F} \mathcal{M}'$ with classifying map a homotopy equivalence (with $\mathcal{N}$, $\mathcal{M}$, $\mathcal{M}'$ symmetric monoidal categories with group structure in the connected components and j, F symmetric monoidal functors which preserve the group structures), we have that the classifying space of $\mathcal{M}$ is of the homotopy type of $B\mathcal{N}$ with the homotopy fiber of the classifying map of F. Recall the functor $\Phi : \mathscr{S}^G_{>0} \longrightarrow \mathscr{S}^G_1$ from Theorem 2.8, where the classifying space of $\mathscr{S}^G_1$ is the Borel construction $\mathbb{N}^{r(G)} \rtimes G$. There is an analog functor for the whole category $\mathscr{S}^G$ to the category of fractions $\mathscr{G}^G = \mathscr{S}^G[\mathscr{S}^{G^{-1}}]$. This has the form

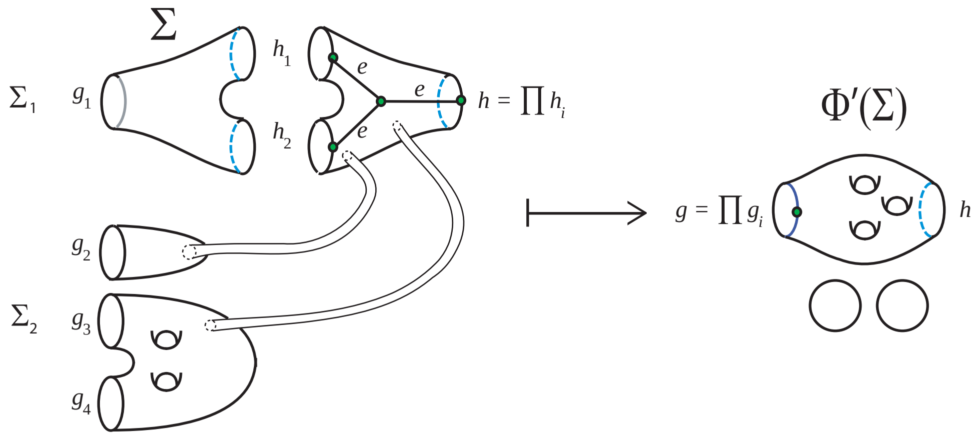
$$\Phi' : \mathscr{S}^G \longrightarrow \mathscr{G}^G .$$

It is important to mention that this functor is not an extension of Φ, for example it sends all the G-cylinders, which start and end in the trivial G-bundle, to the identity. For the minimal surfaces the functor Φ' is defined by,



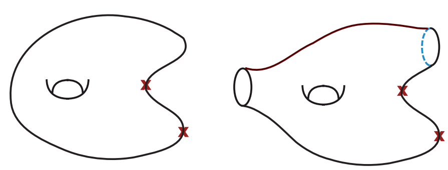
[7] $\mathcal{T}op$ is the category of topological spaces.

For a general G-cobordism we illustrate the definition of Φ' for a particular case as follows



(4.1)

where $\Sigma = \Sigma_1 \sqcup \Sigma_2$, with Σ_1 a G-cobordism in $\mathscr{S}_b^G$ [8] and Σ_2 with empty outgoing boundary. The construction in (4.1), starts by composing Σ_1 with a G-pair of pants (of the form (2.1)) with one exit and the same entries that the exits of Σ_1, and then we glue each connected component of Σ_2 with the last G-pair of pants by a connected sum, where this connected sum is independent of the way we do it because on a surface any two simple contractible closed curves are related by an orientation-preserving homeomorphism[9]. Subsequently, for the resulting connected cobordism, by Cerf theory [**Cer70**], we can find a Morse function with the property that there exists $t \in [0,1]$ such that the inverse image of $[0,t]$ is a G-pair of pants of the form (2.1) and with a circle as the inverse image of t. Finally, we suppose that Σ_2 has c_2 connected components, then we take $\Phi'(\Sigma_2)$ as the class, inside the category of fractions $\mathscr{G}^G$, of the disjoint union of the pre-image of $[t,1]$ by the last Morse function, with a disjoint union of c_2 spheres. This construction is functorial since we can eliminate the cases where we have adjacent critical points of index 0 and 2 [10]. We illustrate some of them in the following picture



.

Therefore, every connected component with empty outgoing boundary can be identified with a unique critical point of index 2.

Let X_G denote the homotopy fiber of Φ'. Consider the sequence of functors $\mathbb{N}^{r(G)} = \mathscr{S}_e^G \hookrightarrow \mathscr{S}^G \longrightarrow \mathscr{G}^G$ with image the fundamental group based at the trivial G-bundle. The composition is just the inclusion $\mathbb{N}^{r(G)} \hookrightarrow \mathbb{Z}^{r(G)}$, and the induced map of classifying spaces is a homotopy equivalence. We conclude with the following result.

Theorem 4.4. *There is a simply connected infinite loop space X_G such that $B\mathscr{S}^G$ is homotopic to the product space $\frac{G}{[G,G]} \times X_G \times T^{r(G)}$.*

We end this section with a categorical description for the space X_G. We recall that for a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ and y an object of $\mathcal{D}$, we define the category $y \diagdown F$ with objects pairs (v,x) with x an object in $\mathcal{C}$ and $v : y \longrightarrow F(x)$ a morphism in $\mathcal{D}$; a morphism $f : (v,x) \longrightarrow (v',x')$ is given by a morphism $f : x \longrightarrow x'$ in $\mathcal{C}$ with $F(f) \circ v = v'$. A celebrated theorem of Quillen, see [**Qui73**], states that for a

[8] Every morphism in $\mathscr{S}_b^G$ has has non-empty final boundary.

[9] Recall that every principal G-bundle over the disk is trivial, consequently, every connected sum extends to the total space of the principal G-bundles.

[10] They locally are of the form $\pm(x^2+y^2)$.

functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, such that for every arrow $y \longrightarrow y'$ in $\mathcal{C}'$, the induced functor $y' \setminus F \longrightarrow y \setminus F$ is a homotopy equivalence, then for any object y of $\mathcal{C}'$ the cartesian square of categories

$$\begin{tikzcd} y \setminus F \arrow[r, "j"] \arrow[d, "f'"'] & \mathcal{C} \arrow[d, "f"] \\ y \setminus \mathcal{C}' \arrow[r, "j'"'] & \mathcal{C}' \end{tikzcd} \qquad \begin{aligned} j(x,v) &= x\,, \\ f'(x,v) &= (fx, v)\,, \\ j'(y',v) &= y'\,. \end{aligned}$$

is homotopy-cartesian[11]. The category $y \setminus \mathcal{C}'$, which corresponds to the comma category associated to the identity functor $1_{\mathcal{C}'}$, has contractible classifying space since it has an initial object, thus the classifying space $B(y \setminus F)$ is of the homotopy type of the homotopy fiber of the classifying map of F. For us the functor $\Phi' : \mathscr{S}^G \longrightarrow \mathscr{G}^G$ has a groupoid as target category which implies the assumption of the theorem of Quillen. As a consequence, the space X_G is of the homotopy type of the classifying space $B(e/\Phi)$, where e represents the trivial G-bundle.

5. An approximation to the number of subgroups of a finite group

An unsolved problem in geometric group theory is to give an explicit formula for the number of subgroups of a (finite) non-abelian group. Even for abelian groups, this is a complicated task, see [**T1̆0, C0̆4**]. There are families of groups which admit a nice description of this number. For example the number of divisors of the integer n, denoted by $\tau(n)$, is the number of subgroups of the cyclic group $\mathbb{Z}_n$. For the Dihedral group D_{2n}, Stephan A. Cavior in [**Cav75**], proved that the number of subgroups is given by $\tau(n) + \sigma(n)$, where $\sigma(n)$ is the sum of the divisors. Similarly, for the dicyclic groups Dic_n, the number of subgroups of Dic_n coincides with $\tau(2n) + \sigma(n)$. For all the groups of order less than thirty, a list of the groups and the number of their subgroups is presented by G. A. Miller in [**Mil40**].

We prove below that for the cyclic group $G = \mathbb{Z}_n$, with n a positive integer, the number $r(\mathbb{Z}_n)$ coincides with the number of subgroups of $\mathbb{Z}_n$, or $\tau(n)$. Thus at first sight we might think that this could be the case for any arbitrary finite group. But by a computational implementation with [**MAT**], we find that the first counterexample is $\mathbb{Z}_2^3$ for the abelian groups, and the dihedral group D_{12} for the nonabelian groups, see table (5.5). In addition, we observe that the more the group G splits, then the number $r(G)$ is far to the number of subgroups of G. Thus it would be interesting to think if the number $r(G)$ associated to a simple group coincides with its number of subgroups. In addition, it is interesting to observe that the number $r(G)$ associated to sequences of groups of the form $G = \mathbb{Z}_2^n$. Indeed, with the help of [**oei**], we found that the number $r(\mathbb{Z}_2^n)$ follows the sequence $2, 5, 15, 51, 187, 715, ...$ which writes as $(2^n + 1)(2^{n-1} + 1)/3$. To my knowledge, see [**oei**], this number represents the dimension of the universal embedding of the symplectic dual polar space, see [**BB03**], or is the number of isomorphism classes of regular four folding coverings of a graph with respect to the identity automorphism, see [**HK93**], or the density of a language L_c with $c = 4$, see [**MR05**]. We prove another variant for this number with the identity

$$r(\mathbb{Z}_2^n) = \frac{(2^n+1)(2^n-1)}{3}\,. \tag{5.1}$$

In this case the number $r(\mathbb{Z}_2^n)$ is delimited by the number of subspaces of the vector space F_2^n, or the Gauss binomial coefficient, which is the number of subgroups of $\mathbb{Z}_2^n$. For any prime number p we generalize the formula (5.1) as follows

$$r(\mathbb{Z}_p^n) = \frac{p^{2n-1} + p^{n+1} - p^{n-1} + p^2 - p - 1}{p^2 - 1}\,.$$

It could be interesting to find a formula for the sequences $r(\mathbb{Z}_k^n)$ for a general positive integer k. Finally, the number $r(G)$ decomposes as a finite sum $r_1(G) + r_2(G) + ...$, where each subindex corresponds to the genus of the generator. We prove below that, for the dihedral and dicyclic groups, the number $r_1(G)$, associated to these groups, coincides with the number of abelian subgroups.

[11] A square is homotopy-cartesian if the canonical map to the homotopy-fiber-product is a homotopy equivalence.

The definition of the positive integer $r(G)$ includes a quotient monoid composed by n-tuples of pairs

$$(g_1, k_1)(g_2, k_2) \cdots (g_n, k_n) \tag{5.2}$$

with $g_i, k_i \in G$, $1 \leq i \leq n$, such that

$$[k_n, g_n][k_{n-1}, g_{n-1}] \cdots [k_1, g_1] = e$$

where $e \in G$ is the identity of the group. We say that the element (5.2) has genus n. These elements make up a monoid with the concatenation as product. The equivalence relation, by which we are going to make the quotient, is generated by some relations which are motivated by the definition of a G-Frobenius algebra, see [**MS06, Tur10, Kau03**]. This relations are the following:

- on generators of genus 1, with the form (g, k), the application of a Dehn twist, see [**FM12**], over a trivial G-cylinder gives the following equation
$$(g, k) \sim (g, h^n k g^m)\,,$$
where $h = kgk^{-1}$ and $n, m \in \mathbb{Z}$; since $[k, g] = e$, then the equation (**??**) simplifies to the following
$$(g, k) \sim (g, kg^m)\,; \tag{5.3}$$
- consider generators of genus 2, with the form $(g_1, k)(g_2, k^{-1})$, the relations (2.4) induce the equation
$$(g_1, k)(g_2, k^{-1}) \sim (g_1^{-1}, k^{-1})((k^{-1})^2 g_2^{-1} k^2, k)\,;$$
- the interchange of critical points of index one for adjacent genus, clockwise and counter-clockwise, induce two equations for generators of genus 2,
$$(g, k)(g', k') \sim ([g, k]g', k')k'(g, k)k'^{-1}\ ,\ (g, k)(g', k') \sim k^{-1}(g', k')k(k^{-1}[k, g]kg, k)\,;$$
- the interchange of the two generators of the fundamental group (non-homotopic to a boundary) of a torus with one boundary circle, correspond to the following identification
$$(g, k) \sim (gkg^{-1}, g^{-1})\,,$$
this equation simplifies, for generators of genus 1, as follows
$$(g, k) \sim (k, g^{-1})\,. \tag{5.4}$$

There are some axioms of a G-Frobenius algebra, for example the twisted commutativity, whose action is trivial due to a cancelation by a Dehn twist. The explicit equations for generators of genus bigger than 2 are combination of the last equations, where we have to be careful with the action of the relations (2.4).

Proposition 5.1. *For a finite group G we have an action of the special linear group $\mathrm{SL}(2, \mathbb{Z}_2)$ into the set of commuting elements $\{(k, g) : [k, g] = e\}$ whose number of orbits is the number of generators of genus 1, i.e. $r_1(G)$.*

Proof. First let G be an abelian group, the special linear group $\mathrm{SL}(2, \mathbb{Z}_2)$ is generated by two matrices,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where the action of the first matrix, form left to right, gives the equation (5.3), while the action of the second matrix gives the equation (5.4). For a general group we only need to prove that the action is well defined, but this follows since the action in any of the two coordinates produces elements given by a product $g^m k^n$ with m, n integers. The proposition follows since g and k commute. □

With the help of [**MAT**] and [**GAP**] we complete the following table which was motivated by [**Mil40**]. This table contains for each group G the number of subgroups, the number of abelian subgroups and the number $r(G)$ associated. We denote by a sum $r_1(G) + r_2(G) + \cdots$ the decomposition of the number $r(G)$ in the components of different genus. Note that in the following table we color the abelian groups with orange and the nonabelian groups with blue. Also note that we color with red the cases when we check

that the number $r(G)$ does not satisfies to be the number of subgroups of G and we color with green the cases when the generator $r_1(G)$ does not satisfies to be the number of abelian subgroups.

(5.5) LIST OF THE GROUPS AND THE NUMBER OF THEIR SUBGROUPS

ORDERS	DESCRIPTION OF THE GROUPS	SUBGROUPS	ABE-SUBGROUPS	GENERATOR'S
4	Cyclic($\mathbb{Z}_4$), $\mathbb{Z}_2^2$	3,5	3,5	3,5
6	$\mathbb{Z}_6$, symmetric(Σ_3)	4,6	4,5	4,5+1
8	$\mathbb{Z}_8$, octic(D_8), quaternion(Q_8)	4,10,6	4,9,5	4,9+1,5+1
	$\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2^3$	8,16	8,16	8,15
9	$\mathbb{Z}_9$, $\mathbb{Z}_3^2$	3,6	3,6	3,7
10	$\mathbb{Z}_{10}$, dihedral(D_{10})	4,8	4,7	4,7+1
12	$\mathbb{Z}_{12}$, tetrahedral(A_4), D_{12}	6,10,16	6,9,13	6,9+1,13+14+_
	Dicyclic(Dic_3), $\mathbb{Z}_2^2 \times \mathbb{Z}_3$	8,10	7,10	7+1,10
14	$\mathbb{Z}_{14}$, D_14	4,10	4,9	4,9+1
15	$\mathbb{Z}_{15}$	4	4	4
16	$\mathbb{Z}_{16}$, Dic_4, D_{16}, ($Q_8 \times \mathbb{Z}_2$)	5,11,19,19	5,8,16,14	5,8+4+_ ,16+4+_,14+10+_
	$\mathbb{Z}_8 \times \mathbb{Z}_2$, $\mathbb{Z}_4^2$, $\mathbb{Z}_4 \times \mathbb{Z}_2^2$, $\mathbb{Z}_2^4$	11,15,27,67	11,15,27,67	11,16,25,51
	Modular group of order 16	11	10	10+1
	Quasihedral of order 16	15	12	12+4+_
	$D_8 \rtimes \mathbb{Z}_2$	35	30	28+_
	$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	23	22	21+_
	$G_{4,4}$	15	14	10+7+_
	$Q_8 \rtimes \mathbb{Z}_2$	23	18	20+_
18	$\mathbb{Z}_{18}$, $\mathbb{Z}_3 \times \mathbb{Z}_6$, D_{18}	6,12,16	6,12,12	6,14,12+_
	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, $\Sigma_3 \times \mathbb{Z}_3$	28,14	15,12	16+_,13+13
20	$\mathbb{Z}_{20}$, $\mathbb{Z}_{10} \times \mathbb{Z}_2$, D_{20}	6,10,22	6,10,19	6,10 ,19+_
	Dic_5, metacyclic	10,14	9,12	9+1, 12+3+_
21	$\mathbb{Z}_{21}$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$	4,10	4,9	4,9+1
22	$\mathbb{Z}_{22}$, D_{22}	4,14	4,13	4,13+4+_
24	$\mathbb{Z}_{24}$,$\mathbb{Z}_2 \times \mathbb{Z}_{12}$,$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$	8,16,32	8,16,32	8,16,30
	$D_8 \times \mathbb{Z}_3$,$Q_8 \times \mathbb{Z}_3$	20,12	18,10	18+_,10+_
	$Sl(2,3)$, $A_4 \times \mathbb{Z}_2$	15,26	13,24	13+_,23+_
	Σ_4, D_{24}, Dic_6	30,34,18	21,24,12	21+3+_ ,24+_ ,12+
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$, $\mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$	54,22	43,19	40+_,19+_
	$\mathbb{Z}_4 \times \Sigma_3$,$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$	26,10	21,9	21+_,9+_
	$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	30	22	22+
25	$\mathbb{Z}_{25}$, $\mathbb{Z}_5^2$	3,8	3,8	3,11
26	$\mathbb{Z}_{26}$, D_{26}	4,16	4,15	4,15+_
27	$\mathbb{Z}_{27}$, $\mathbb{Z}_9 \times \mathbb{Z}_3$, $\mathbb{Z}_3^3$	4,10,28	4,10,28	4,12,40
	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$, $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$	19,10	18,9	22+_,10+_
28	$\mathbb{Z}_{28}$, $\mathbb{Z}_{14} \times \mathbb{Z}_2$, D_{28}, Dic_7	6,10,28,12	6,10,25,11	6,10,25+_ ,11+_

THEOREM 5.2. *For the cyclic groups $\mathbb{Z}_n$, the number $r(\mathbb{Z}_n)$ coincides with the number of subgroups of $\mathbb{Z}_n$.*

PROOF. For the cyclic group $G = \mathbb{Z}_n$ and $(g,k) \in \mathbb{Z}_n \times \mathbb{Z}_n$ we have the division algorithm

$$\begin{aligned} k &= q_1 g + r_1 \\ -g &= q_2 r_1 + r_2 \\ r_1 &= q_3 r_2 + r_3 \\ &\vdots \\ \pm r_{m-1} &= q_{m+1} r_m \end{aligned}$$

with $0 \leq |r_{m-1}| < ... < |r_1| < |k|$. Thus by the equations (5.3) and (5.4) we have the following sequence of identifications

$$\begin{aligned} (g,k) &\sim (g,r_1) \sim (r_1,-g) \sim (r_1,r_2) \sim (r_2,-r_1) \sim (r_2,-r_3) \sim \cdots \sim (r_{m-1},r_{m-2}) \sim (r_{m_1},-qr_{m-1}) \\ &\sim (r_{m-1},0)\,. \end{aligned}$$

Since $\sim$ is an equivalence relation and $(p,0) \sim (np,mp)$ for some $n,m \in \mathbb{Z}$ (and similar for q), then $(p,0) \sim (q,0)$ if and only if $\langle p \rangle = \langle q \rangle$. Therefore, $r(\mathbb{Z}_n)$ is the number of subgroups of $\mathbb{Z}_n$. □

THEOREM 5.3. *The numbers $r_1(D_{2n})$ and $r_1(Dic_n)$ coincide with the number of abelian subgroups of D_{2n} and Dic_n respectively.*

PROOF. A presentation for the dihedral group is as follows

$$D_{2n} = \langle r,s | r^n = s^2 = 1, sr = r^{-1}s \rangle\,.$$

We know that any subgroup of D_{2n} is subgroup of $\langle r \rangle$ or is of the form $\langle sr^i, r^m \rangle$ for $m = n/d$ with d a divisor of n. Thus any subgroup of D_{2n} is cyclic or is generated by two elements. We can find a correspondence between the abelian subgroups of D_{2n} and the generators of genus 1 as follows. If the subgroup is cyclic with generator a, then we assign the class of the pair $(a,0)$. If the subgroup is generated by two elements, say a and b, then we assign the class of the pair (a,b). This assignment is clearly surjective and, by the application of the equations (5.3) and (5.4), it is injective.

A presentation for the dicyclic group is as follows

$$Dic_n = \langle r,s | r^{2n} = 1, r^n = s^2, sr = r^{-1}s \rangle\,.$$

The arguments of the last paragraph work, where we consider the pair (r^m, sr^i) as a representant for the group $\langle sr^i, r^m, r^n \rangle$. □

THEOREM 5.4. *For the group $\mathbb{Z}_p^n$, with p a prime number, we have the identity*

$$r(\mathbb{Z}_p^n) = \frac{p^{2n-1} + p^{n+1} - p^{n-1} + p^2 - p - 1}{p^2 - 1}\,.$$

PROOF. For $r_p^n := r(\mathbb{Z}_p^n)$, let $F(n)$ be the number $r_p^{n+1} - r_p^n$. We will prove that

$$F(n) = p^{n-1}(p^n + p - 1)\,. \tag{5.6}$$

Since $r_p^n = (r_p^n - r_p^{n-1}) + (r_p^{n-1} - r_p^{n-2}) + ... + (r_p^3 - r_p^2) + (r_p^2 - r_p^1) + r_p^1$, where $r_p^1 = 2$ by Theorem 5.2. Thus $r_p^n = p^{n-2}(p^{n-1} + p - 1) + p^{n-3}(p^{n-2} + p - 1) + ... + p(p^2 + p - 1) + (p + p - 1) + 2$ and as a

consequence we have the following equations

$$
\begin{aligned}
r_p^n &= \sum_{i=0}^{n-2} p^{2i+1} + (p-1)\sum_{i=0}^{n-2} p^i + 2 \\
&= p\frac{(p^2)^{n-1}-1}{p^2-1} + (p-1)\frac{p^{n-1}-1}{p-1} + 2 \\
&= \frac{p^{2n-1}-p+(p^{n-1}-1)(p^2-1)}{p^2-1} \\
&= \frac{p^{2n-1}+p^{n+1}-p^{n-1}+p^2-p-1}{p^2-1}
\end{aligned}
$$

We will prove the formula (5.6), by induction, and we observe the following

$$F(n) = pF(n-1) + p^{2n-2}(p-1)\,.$$

Let $\mathrm{Mat}(n \times 2, \mathbb{Z}_p)$ be the matrices with coefficients in $\mathbb{Z}_p$, then the equations (5.3) and (5.4) are given by the generators of the special linear group $\mathrm{SL}(2, \mathbb{Z}_p)$ as follows

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}.$$

Thus the number r_p^n is the same as the number of orbits of the quotient of $\mathrm{Mat}(n \times 2, \mathbb{Z}_p)$ by the group $\mathrm{SL}(2, \mathbb{Z}_p)$. By definition, $F(n)$ consists of elements in $\mathrm{Mat}(n \times 2, \mathbb{Z}_p)$ such that the last column is different than zero. There are three cases to consider:

(1) The representatives of the classes have zeros in the coordinate n, i.e. the matrix has the form
$$\begin{pmatrix} \cdots & 0 & i \\ \cdots & 0 & j \end{pmatrix}$$
with $i, j \neq 0$, at least one, and for this case the number of classes is $F(n-1)$.

(2) The representatives of the classes have zeros in the second row for the last two columns, i.e. the matrix has the form
$$\begin{pmatrix} \cdots & i & j \\ \cdots & 0 & 0 \end{pmatrix}$$
with $i \neq 0$ and $j \neq 0$. For these elements the stabilizer group is the same, before and after erasing the last column, so we have $(p-1)F(n-1)$ classes, where we multiply by $p-1$ since we can not take the zero value for j.

(3) The last case is composed by classes with representative of the form
$$\begin{pmatrix} \cdots & i & 0 \\ \cdots & 0 & j \end{pmatrix} \tag{5.7}$$
with $i \neq 0$ and $j \neq 0$. Every stabilizer of an element of the form (5.7), has to be the identity, then the classes have cardinality the order of $\mathrm{SL}(2, \mathbb{Z}_p)$ which is $p(p^2-1)$. For the calculation of the number of classes we consider all the matrix of the form (5.7), inside $\mathrm{Mat}(n \times 2, \mathbb{Z}_p)$; for this we multiply $p^{n-1}p^{n-1}$, given by the first $n-1$ columns, with the index $\left|\frac{\mathrm{GL}(2,\mathbb{Z}_p)}{\mathrm{SL}(2,\mathbb{Z}_p)}\right| = p-1$ since the last columns represents an element in $\mathrm{Gl}(2, \mathbb{Z}_p)$. Therefore, the classes are $p^{2n-2}(p-1)$. The sum of the numbers associated to these three cases finalize the proof of the theorem.

□

References

[BB03] A. Blokhuis and A.E. Brouwer, *The universal embedding dimension of the binary symplectic dual polar space*, Discrete Mathematics **264, Issues 1-3** (2003), 3–11.

[BT99] Mark Brightwell and Paul Turner, *Representations of the homotopy surface category of a simply connected space*, http://arxiv.org/abs/math/9910026, October 1999.

[Cav75] S. R. Cavior, *The subgroups of the dihedral group*, Math. Mag **48** (1975), 107.

[Cer70] Jean Cerf, *La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie*, Inst. Hautes Études Sci. Publ. Math. **39** (1970), 5–173.

[CŎ4] Grigore Călugăreanu, *The total number of subgroups of a finite abelian group*, Scientiae Mathematicae Japonicae **60, No. 1** (2004), 157–167.

[EH60] Beno Eckmann and P.J. Hilton, *Operators and cooperators in homotopy theory*, Mathematische Annalen **141** (1960), 1–21.

[FM12] Benson Farb and Dan Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, no. 49, Princeton University Press, 2012.

[GAP] GAP, *Groups, algorithms, programming*, www.gap-system.org.

[GM03] Sergei Gelfand and Yuri Manin, *Methods of homological algebra*, second edition ed., Springer Monographs in Mathematics, Springer-Verlag, 2003.

[GMTW] Søren Galatius, Ib Madsen, Ulrike Tillmann, and Michael Weiss, *The homotopy type of the cobordism category*, Acta Mathematica **202**, no. 2, 195–23.

[GZ67] Peter Gabriel and Michel Zisman, *Calculus of fractions and homotopy theory*, Springer-Verlag New York Inc., 1967.

[HK93] Sungpyo Hong and Jin Ho Kwak, *Regular fourfold coverings with respect to the identity automorphism*, Journal of Graph Theory **17** (1993), 621–627.

[Kau03] Ralph M. Kaufmann, *Orbifolding frobenius algebras*, International Journal of Mathematics **14, Issue: 6** (2003), 573–617.

[MAT] MATLAB, *Matrix laboratory*, www.mathworks.de/company/worldwide/.

[May72] J.P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics **271** (1972).

[May74] ———, *E_∞ spaces, group completion, and permutative categories*, Lecture Notes London Math. Soc. **11** (1974), 61?92.

[Mil40] G. A. Miller, *Subgroups of the groups whose order are below thirty*, Proceedings of the National Academy of Sciences of the United Stated of America **26(8)** (1940), 500–502.

[MR05] Nelma Moreira and Rogério Reis, *On the density of languages representing finite set partitions*, Journal of Integer Sequences **8** (2005), 1–11.

[MS06] Gregory W. Moore and Graeme Segal, *D-branes and k-theory in 2d topological field theory*, http://arxiv.org/abs/hep-th/0609042, September 2006.

[MW07] Ib Madsen and Michael Weiss, *The stable moduli space of riemann surfaces: Mumford's conjecture*, Annals of Mathematics **165** (2007), 843–941.

[oei] *The on-line encyclopedia of integer sequences.*

[Pha10] David N. Pham, *Groupoid frobenius algebras*, Communications in Contemporary Mathematics **12** (2010), 939–952.

[Qui73] Daniel Quillen, *Higher algebraic k-theory:* I*, in: Algebraic k-theory i*, Lect. Notes Math. **341** (1973), 77–139.

[Seg68] Graeme Segal, *Classifying spaces and spectral sequences*, Publications Math?matiques de l'IH?S **34** (1968), 105–112.

[Seg74] ———, *Categories and cohomology theories*, Topology **13** (1974), 293–312.

[Seg11] Carlos Segovia, *The classifying space of the 1+1 dimensional g-cobordism category*, Ph.D. thesis, CINVESTAV, 2011.

[Til96] Ulrike Tillmann, *The classifying space of the 1+1 dimensional cobordism category*, J. für die reine und angewandte Mathematik **479** (1996), 67–75.

[Tur10] Vladimir Turaev, *Homotopical field theory in dimension 2 and group-algebras*, European Mathematical Society (EMS) **10** (2010).

[TĬ0] Marius Tărnăuceanu, *An arithmetic method of counting the subgroups of a finite abelian group*, Bull. Math. Soc. Sci. Math. Roumanie **53(101) No. 4** (2010), 373?386.

Mathematisches Institut, Universität Heidelberg, Deutschland

E-mail address: `csegovia@mathi.uni-heidelberg.de`